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QUALITATIVE ASPECTS OF THE MOTION OF RIGID BODIES WITH LIQUID-FILLED TOROIDAL CAVITIES

J. H. Giese

Department of the Army Project No. 503-06-002
Ordnance Management Structure Code No. 5010.11.812

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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ABSTRACT

For a rigid body subject to no moments the differential equations for the angular velocity can be solved independently of the remaining equations of motion. The integral curves are intersections of the energy and angular momentum ellipsoids, which have common centers and principal axes. In general, there are <u>four</u> types of closed integral curves. It is well known that if the solid contains a cavity that is topologically equivalent to (i.e., continuously deformable into) the interior of a sphere and is completely filled with non-viscous incompressible fluid, the properties mentioned above remain valid. However, if the cavity is topologically equivalent to the interior of a torus, the fact that the fluid may have a non-vanishing circulation, Γ , on certain paths creates new possibilities. The angular velocity integral curves are still intersections of ellipsoids with parallel principal axes, but one of the centers has been displaced through a distance that depends on the parameter Γ . If $\Gamma = 0$ there are generally <u>four</u> types of closed integral curves; <u>five</u> for "weak" circulation; <u>three</u> for "intermediate" $|\Gamma|$; and <u>one</u> for "strong" $|\Gamma|$. The qualitative nature of the integral curves for solids with

cavities of greater topological complexity has also been analyzed. The number of distinct types of behavior is surprisingly limited and is, in fact, still closely akin to that of bodies with toroidal cavities.

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1. INTRODUCTION

To determine the motion of a liquid-filled solid is an important and difficult problem in dynamics. In actual applications the liquid is viscous and need not completely fill the cavity in which it is stored. However, when one attempts to attack this problem, a natural model with which to begin is that of a cavity completely filled with an incompressible, non-viscous liquid in irrotational motion.

If the cavity is topologically spherical, (i.e., homeomorphic to the interior of a sphere), then in this model the liquid filling introduces no dynamic novelties. The differential equations of motion of the solid-liquid system can simply be considered to be those of a solid with a different mass and different inertial matrix. Among its other shortcomings there is no possibility for including in this unsteady irrotational flow a "spin" or general circulation, such as one would expect to obtain if, for example, the container itself is spinning.

If the cavity is topologically equivalent to the interior of a torus, it becomes possible to have unsteady irrotational flow with circulation, Γ. Unfortunately Γ must be constant, a fact which seriously restricts prospective application of the following theory. Nevertheless it seems worthwhile to investigate the influence of circulation on the motion of the composite solid-liquid system. For this purpose, the most natural and easiest problem to consider is the classical one of motion under no forces and moments.

If we consider a system with no special symmetry, the results may be summarized as follows. As will be shown later, the range of Γ^2 must be divided into three intervals with end points 0, Γ_2^2, Γ_1^2, and ∞. For $\Gamma = 0$ we obtain the classical result that there are six possible steady states of rotation, four of them stable, the other two unstable. Roughly speaking, there are four periodic types of angular motion, each centered about one of the stable steady states. For weak circulation, $0 < \Gamma^2 < \Gamma_2^2$, there are still six steady states, four of them stable.

Now, however, in addition to four previously mentioned types of periodic motion, each centered about one of the stable steady states, there is now a fifth type of periodic motion centered about two stable steady states. For intermediate circulation, $\Gamma_2^2 < \Gamma^2 < \Gamma_1^2$, there are four steady states, three of them stable, and three types of periodic motion. Finally, for strong circulation, $\Gamma_1^2 < \Gamma^2$, there are two stable steady states, and one type of periodic motion.

Certain types of degeneracy are associated with appearances of multiple characteristic roots of a matrix involved in the theory, or special orientations of an "axis" vector associated with the cavity. An important example of such degeneracy occurs for axisymmetric liquid-solid systems.

If the cavity is of topological genus N, i.e. topologically equivalent to the interior of a sphere with N handles, the analysis of the possible motions can be performed by an extension of the methods developed for the discussion of toroidal cavities (N = 1). With a cavity of genus N we associate N "axis" vectors, of which not more than three can be linearly independent, of course. This yields the conclusion that there are essentially four types of motion for bodies with liquid-filled cavities of genus N, depending on the number, between zero and three, of independent "axis" vectors. Two types correspond to topologically spherical or toroidal cavities (N = 0 or 1). We conjecture, without proof, that the other two types correspond to cavities of genera N = 3 or 4. The measure of arbitrariness available for the choice of circulations for cavities of genus $N \geq 4$ merely serves to provide a greater variety of assignments of circulation that will produce the same motions as a body with a cavity of genus $N \leq 3$.

2. FLOWS WITH MOVING BOUNDARIES

The most commonly discussed problem that involves rigid bodies and a non-viscous incompressible fluid concerns the motion of a finite solid through an infinite region filled with fluid[2,3]. The method used for this classical problem requires only slight modification to adapt it to develop the equations of motion of a finite rigid body that contains a cavity completely filled with fluid. However, since our purpose in the following discussion is to emphasize the influence of circulation and since the derivation is comparatively short, for the sake of completeness and clarity we shall reproduce it here.

First choose a moving system of rectangular coordinates relative to axes rigidly attached to the body and with origin at the center of mass of the combined mass of body and fluid. At any time t also choose coordinates (x,y,z) relative to a system of axes fixed in space that coincides with the instantaneous position of the moving axes. At this instant the position of any point in the body or the fluid can be specified by its coordinate vector $\underline{x} = (x,y,z)$. Also at time t let $\underline{u}(t) = (u_x, u_y, u_z)$ be the velocity of the moving origin, and let $\underline{w}(t) = (w_x, w_y, w_z)$ be the angular velocity of the moving system relative to the fixed axes.

Now suppose the body contains a cavity filled with incompressible nonviscous fluid of density ρ_L. Let V_L be volume occupied by the cavity, and S_L its boundary. If we assume that the motion of the liquid was started impulsively from rest, then by Kelvin's theorem it will be irrotational. Accordingly, the velocity of the liquid can be expressed as

$$U_L(\underline{x}, t) = \nabla \Phi(\underline{x}, t) \tag{2.1}$$

for some velocity-potential function Φ such that

$$\nabla^2 \Phi = 0 \quad \text{in } V_L \tag{2.2}$$

and subject to the condition that on S_L the normal component of the velocity of the fluid relative to that of the boundary vanishes. Let X_L be any point of S_L and $\underline{n}$ the corresponding unit normal directed <u>into</u> V_L. Then we must have

$$\partial\Phi/\partial n = \underline{n} \cdot \nabla\Phi(\underline{x}, t) = \underline{n} \cdot \left[\underline{u}(t) + \underline{w}(t) \times \underline{x} \right] \text{ on } S_L. \quad (2.3)$$

Since $\underline{n} = \underline{n}(\underline{x})$ at time t, tentatively choose

$$\Phi(\underline{x}, t) = \underline{u}(t) \cdot \underline{\phi}(\underline{x}) + \underline{w}(t) \cdot \underline{\sigma}(\underline{x}) \quad (2.4)$$

Then (2.2) and (2.3) will be satisfied if we choose $\underline{\phi}(\underline{x})$ and $\underline{\sigma}(\underline{x})$ to be single-valued solutions of

$$\nabla^2 \underline{\phi} = 0, \qquad \nabla^2 \underline{\sigma} = 0 \qquad \text{in } V_L \quad (2.5)$$

and

$$\left.\begin{aligned} \partial\underline{\phi}/\partial n &= \underline{n} \cdot \nabla\underline{\phi} = \underline{n} \\ \partial\underline{\sigma}/\partial n &= \underline{n} \cdot \nabla\underline{\sigma} = \underline{x} \times \underline{n} \end{aligned}\right\} \text{ on } S_L \quad (2.6)$$

If $\underline{\phi}_L$ is topologically equivalent to the interior of a sphere, then Φ, $\underline{\phi}$, and $\underline{\sigma}$ must all be single-valued, and as solutions of Neumann problems they must be unique except for additive constants. If, however, V_L is topologically equivalent to the interior of a torus, and if we overlook the question how one would create a general circulation in the cavity, and thus abandon the impulsive start from rest, then Φ need no longer be single-valued. This can be seen by considering the circulation

$$\Gamma(C) = \int_C \underline{U}_L(\underline{x}, t) \cdot d\underline{x} = \int_C d\Phi \quad (2.7)$$

about any simply-closed path C in V_L. Now form $\Gamma(C')$ for any other simply closed path C' which can be continuously deformed into C without crossing S_L. If under this deformation the sense in which C' is traversed in $\Gamma(C')$ corresponds to the sense of C in $\Gamma(C)$, then by Stokes' theorem $\Gamma(C') = \Gamma(C)$; otherwise $\Gamma(C') = -\Gamma(C)$. If, in particular C can

be continuously deformed into a point within V_L, then $\Gamma(C) = 0$. On the other hand, if C loops once, and C' loops N times in the same sense about the hole of the torus, then $\Gamma(C') = N\,\Gamma(C)$. Thus, for all single loops traversed in the same sense we get the same circulation $\Gamma(C) = \Gamma$. If $\Gamma \neq 0$, (2.7) implies that Φ must be multiple-valued. Accordingly, let us modify (2.4) to the form

$$\Phi(\underline{x}, t) = \underline{u}(t) \cdot \underline{\phi}(\underline{x}) + \underline{w}(t) \cdot \underline{\sigma}(\underline{x}) + \Gamma\,\tau(\underline{x}) \tag{2.8}$$

where in addition to (2.5) and (2.6) we require

$$\nabla^2\tau = 0 \qquad \text{in } V_L \tag{2.9}$$

$$\partial\tau/\partial n = \underline{n} \cdot \nabla\tau = 0 \qquad \text{on } S_L \tag{2.10}$$

and though the components of $\nabla\tau$ are single-valued, τ increases by unity when a closed loop about the hole of the torus is traversed once in an arbitrarily selected positive sense. By Bernoulli's theorem

$$-\,p/\rho_L = \partial\Phi/\partial t + 0.5\,(\nabla\Phi)^2 + g\underline{G}(t) \cdot \underline{x} + F(t) \tag{2.11}$$

where $\underline{G}(t)$ is a unit vector parallel to the gravitational field and $F(t)$ is some function of t. Since the pressure, p, must be single-valued in V_L, then $d\Gamma/dt = 0$, i.e. Γ <u>must be a constant</u>.

Hereafter it will suffice to consider (2.8) without special reference to (2.4), since results for topologically spherical cavities can be deduced by merely setting $\Gamma = 0$ in the following discussion.

For general cavities V_L of finite extent, which are not necessarily even topologically toroidal,

$$\underline{\phi}(\underline{x}) = \underline{x} \tag{2.12}$$

satisfies (2.5) and (2.6). The discussion of $\underline{\sigma}(\underline{x})$ and $\tau(\underline{x})$, however, cannot be continued without specializing V_L. This will be done for axisymmetric cavities in Section 7.

3. LINEAR AND ANGULAR MOMENTA OF THE LIQUID

To formulate the equations of motion of our composite solid-liquid system we shall require the linear and angular momenta of the liquid. Let $\underline{\xi}_L(t)$ be the linear momentum, and $\underline{k}$ an arbitrary constant vector. Then

$$\underline{\xi}_L \cdot \underline{k} = \int_{V_L} \rho_L \underline{k} \cdot \nabla\Phi \, dV = J_1 + J_2 + J_3 \qquad (3.1)$$

where, in accordance with (2.8) and (2.12), the scalars J are defined below. First

$$J_1 = \int_{V_L} \rho_L \underline{k} \cdot \underline{u} \, dV = M_L \, \underline{k} \cdot \underline{u} \qquad (3.2)$$

where M_L is the total mass of the liquid. Next

$$J_2/\rho_L = \int_{V_L} \underline{k} \cdot \nabla (\underline{w} \cdot \underline{\sigma}) \, dV = \int_{V_L} \nabla (\underline{k} \cdot \underline{x}) \cdot \nabla (\underline{w} \cdot \underline{\sigma}) \, dV$$

By Green's theorem and (2.6)

$$\begin{aligned} J_2/\rho_L &= -\int_{S_L} \underline{k} \cdot \underline{x} \; \partial(\underline{w} \cdot \underline{\sigma})/\partial n \, dS \\ &= -\int_{S_L} (\underline{k} \cdot \underline{x}) (\underline{w} \times \underline{x} \cdot \underline{n}) \, dS \\ &= \int_{V_L} \nabla \cdot \left[(\underline{k} \cdot \underline{x}) \, \underline{w} \times \underline{x} \right] dV \\ &= \int_{V_L} \underline{w} \times \underline{x} \cdot \underline{k} \, dV \end{aligned}$$

Thus

$$J_2 = M_L \, \underline{w} \times \underline{x}_L \cdot \underline{k} \qquad (3.3)$$

where $\underline{x}_L$ is the center of mass of the liquid. Finally

$$\begin{aligned} J_3/\rho_L &= \int_{V_L} \Gamma \, \underline{k} \cdot \nabla \tau \, dV = \Gamma \int_{V_L} \nabla (\underline{k} \cdot \underline{x}) \cdot \nabla \tau \, dV \\ &= -\Gamma \int_{S_L} (\underline{k} \cdot \underline{x}) \; \partial\tau/\partial n \, dS \end{aligned}$$

whence by (2.10)

$$J_3 = 0 \tag{3.4}$$

Since $\underline{k}$ was arbitrary, these results imply

$$\underline{\xi}_L = M_L (\underline{u} + \underline{w} \times \underline{x}_L) \tag{3.5}$$

Next, let $\eta_L(t)$ be the angular momentum of the liquid. Then

$$\underline{\eta}_L = \int_{V_L} \rho_L \underline{x} \times \nabla\Phi \, dV = \underline{J}_4 + \underline{J}_5 + \underline{J}_6 \tag{3.6}$$

for the following choices of vectors $\underline{J}$. First

$$\underline{J}_4 = \int_{V_L} \rho_L \underline{x} \times \underline{u} \, dV = M_L \underline{x}_L \times \underline{u} \tag{3.7}$$

Next, by Green's theorem and (2.6)

$$\begin{aligned} \underline{J}_5/\rho_L &= \int_{V_L} \underline{x} \times \nabla (\underline{w} \cdot \underline{\sigma}) \, dV = -\int_{V_L} \nabla \times \left[(\underline{w} \cdot \underline{\sigma}) \, \underline{x} \right] dV \\ &= -\int_{S_L} (\underline{w} \cdot \underline{\sigma}) \, \underline{x} \times \underline{n} \, dS = -\int_{S_L} (\underline{w} \cdot \underline{\sigma}) \, \partial\underline{\sigma}/\partial n \, dS \\ &= \int_{V_L} \nabla (\underline{w} \cdot \underline{\sigma}) \cdot \nabla \underline{\sigma} \, dV \end{aligned} \tag{3.8}$$

Finally

$$\underline{J}_6/\rho_L \Gamma = \int_{V_L} \underline{x} \times \nabla \tau \, dV = -\int_{V_L} \nabla \times (\tau \underline{x}) \, dV$$

By means of some surface S^* bounded by a closed curve C^* on S_L change V_L into a topologically spherical region in which τ is single valued. Let $\underline{n}_i$ and τ_i denote the inward unit normal and value of τ on the "initial" side of S^*, and let $\underline{n}_f = -\underline{n}_i$ and $\tau_f = \tau_i + 1$ be the corresponding functions at the same points on the opposite or "final" side of S^*. Then

$$\underline{J}_6/\rho_L \Gamma \equiv \underline{J}^*/\rho_L = -\int_{S_L} (\underline{x} \times \underline{n}) \, \tau \, dS + \int_{S^*} (\underline{x} \times \underline{n}_i) \, dS \tag{3.9}$$

Later, in order to simplify the equations of motion, we shall make transformations of coordinates that can be most easily motivated by exploiting the relation between $\underline{\xi}_L$, $\underline{\eta}_L$, and the kinetic energy T_L of the liquid[2,3]. We have

$$2T_L = \int_{V_L} \rho_L \, (\nabla\Phi)^2 \, dV = J_7 + J_8 + J_9 \tag{3.10}$$

where

$$J_7 = \int_{V_L} \rho_L \, (\underline{u} + \nabla \, \underline{w} \cdot \underline{\sigma})^2 \, dV \tag{3.11}$$

$$J_8 = 2\int_{V_L} \rho_L \, \Gamma (\underline{u} + \nabla \underline{w} \cdot \underline{\sigma}) \cdot \nabla \tau \, dV$$

$$J_9 = \Gamma^2 \int_{V_L} (\nabla\tau)^2 \, dV \equiv \Gamma^2 J_9^* \tag{3.12}$$

By Green's theorem

$$J_8 = 2\rho_L \Gamma \int_{V_L} \nabla (\underline{u} \cdot \underline{x} + \underline{w} \cdot \underline{\sigma}) \cdot \nabla \tau \, dV$$

$$= -2\rho_L \Gamma \int_{S_L} (\underline{u} \cdot \underline{x} + \underline{w} \cdot \underline{\sigma}) \, \partial\tau/\partial n \, dS$$

whence by (2.10)

$$J_8 = 0 \tag{3.13}$$

Now observe that

$$\partial T_L/\partial \underline{u} = \int_{V_L} \rho_L \nabla \Phi \, dV = \underline{\xi}_L \tag{3.14}$$

Also

$$\partial T_L/\partial \underline{w} = \int_{V_L} \rho_L \nabla\Phi \cdot \nabla \, \underline{\sigma} \, dV = \int_{V_L} \rho_L \, (\underline{u} + \nabla \, \underline{w} \cdot \underline{\sigma} + \Gamma \nabla\tau) \cdot \nabla \underline{\sigma} \, dV$$

If in J_8 we set $\underline{u} = 0$ we obtain for any $\underline{w}$

$$\int_{V_L} \rho_L \, \Gamma \nabla\tau \cdot \nabla \, \underline{\sigma} \, dV = 0$$

Thus by Green's theorem and (2.6)

$$(\partial T_L/\partial \underline{w})/\rho_L = \int_{V_L} \nabla (\underline{u} \cdot \underline{x} + \underline{w} \cdot \underline{\sigma}) \cdot \nabla \underline{\sigma} \; dV$$

$$= -\int_{S_L} (\underline{u} \cdot \underline{x} + \underline{w} \cdot \underline{\sigma}) \, \underline{x} \times \underline{n} \; dS$$

$$= -\int_{V_L} \nabla \times \left[(\underline{u} \cdot \underline{x} + \underline{w} \cdot \underline{\sigma}) \, \underline{x} \right] dV$$

$$= \int_{V_L} \underline{x} \times \nabla \, (\Phi - \Gamma \tau) \; dV$$

Hence

$$\underline{\eta}_L = \partial T/\partial \underline{w} + \Gamma \, \underline{J}^* \tag{3.15}$$

where we shall call $\underline{J}^*$ the <u>axis-vector</u> associated with τ.

It will prove convenient to write T_L in matrix notation. First observe that

$$J_7 = M_L \, \underline{u}^2 + 2 \int_{V_L} \rho_L \, u \cdot \nabla \, \underline{w} \cdot \underline{\sigma} \; dV + \int_{V_L} \rho_L \, (\nabla \, \underline{w} \cdot \underline{\sigma})^2 \; dV$$

Thus, if we interpret $\underline{u}$ and $\underline{w}$ as column vectors, then

$$2T_L = M_L \underline{u}^T \, I\underline{u} + 2\underline{u}^T B_L \, \underline{w} + \underline{w}^T C_L \, \underline{w} + J_9^* \, \Gamma^2 \tag{3.16}$$

where I is the 3 x 3 identity matrix, B_L and C_L are matrices with constant elements, and

$$C_L^{\,T} = C_L \tag{3.17}$$

where the superscript T denotes "transpose", and J_9^* is a constant. Since, by repeatedly used manipulations

$$\underline{u}^T B_L\, \underline{w}/\rho_L = \int_{V_L} \underline{u} \cdot \nabla\, \underline{w} \cdot \underline{\underline{\sigma}}\ dV = - \int S_L\, (\underline{u} \cdot \underline{x})\ \partial\, (\underline{w} \cdot \underline{\sigma})/\partial n\ dS$$

$$= - \int_{S_L} (\underline{u} \cdot \underline{x})\ \underline{w} \cdot \underline{x} \times \underline{n}\ dS$$

$$= \underline{w} \cdot \int_{V_L} \nabla \times (\underline{u} \cdot \underline{x})\ \underline{x}\quad dV$$

$$= \int_{V_L} (\underline{w} \cdot \underline{u} \times \underline{x})\ dV$$

$$\underline{u}^T B_L\, \underline{w} = M_L\, \underline{w} \cdot \underline{u} \times \underline{x}_L \tag{3.18}$$

4. LINEAR AND ANGULAR MOMENTA OF THE SOLID

Let V_S be the volume occupied by a solid of mass M_S, density ρ_S, and with center of mass at $\underline{x}_S$. Then the linear momentum of the solid is

$$\underline{\xi}_S = \int_{V_S} \rho_S (\underline{u} + \underline{w} \times \underline{x})\, dV = M_S (\underline{u} + \underline{w} \times \underline{x}_S) \tag{4.1}$$

and its angular momentum is

$$\begin{aligned} \underline{\eta}_S &= \int_{V_S} \rho_S\, \underline{x} \times (\underline{u} + \underline{w} \times \underline{x})\, dV \\ &= M_S\, \underline{x}_S \times \underline{u} + \int_{V_S} \rho_S \left[\underline{x}^2 \underline{w} - (\underline{w} \cdot \underline{x})\, \underline{x} \right] dV \end{aligned} \tag{4.2}$$

Its kinetic energy T_S can be calculated from

$$\begin{aligned} 2T_S &= \int_{V_S} \rho_S (\underline{u} + \underline{w} \times \underline{x})^2\, dV \\ &= M_S u^2 + 2M_S \underline{u} \cdot \underline{w} \times \underline{x}_S + \int_{V_S} \rho_S \left[\underline{x}^2 \underline{w}^2 - (\underline{x} \cdot \underline{w})^2 \right] dV \end{aligned} \tag{4.3}$$

Clearly

$$\underline{\xi}_S = \partial T_S / \partial \underline{u} \tag{4.4}$$

$$\underline{\eta}_S = \partial T_S / \partial \underline{w} \tag{4.5}$$

In matrix notation T_S takes the form

$$2T_S = M_S\, \underline{u}^T I \underline{u} + 2\underline{u}^T B_S\, \underline{w} + \underline{w}^T C_S\, \underline{w} \tag{4.6}$$

where B_S and C_S are constant matrices, and

$$C_S^T = C_S \tag{4.7}$$

$$\underline{u}^T B_S\, \underline{w} = M_S\, \underline{u} \cdot \underline{w} \times \underline{x}_S \quad . \tag{4.8}$$

5. EQUATIONS OF MOTION

For the composite liquid-solid system the total kinetic energy

$$T = 0.5\, M\underline{\underline{u}}^T \underline{\underline{I}}\underline{u} + \underline{\underline{u}}^T B\, \underline{w} + 0.5\, \underline{w}^T C\, \underline{w} + \Gamma^2 \; J_9^{*} \qquad (5.1)$$

where $M = M_L + M_S$ is the total mass, $B = B_L + B_S$, and $C = C_L + C_S$ are constant vectors, and

$$C^T = C \qquad (5.2)$$

Since T must be a positive definite quadratic form in Γ^2 and the components of $\underline{u}$ and $\underline{w}$, then as a matter of fact C must also be positive definite (3.18) and (4.8) imply

$$\underline{u}^T B \underline{w} = \underline{u} \cdot \underline{w} \times \left[M_L \underline{x}_L + M_S \underline{x}_S \right] \qquad (5.3)$$

Hereafter we shall assume that the origin is at the center of mass of the composite system. Thus $M_L \underline{x}_L + M_S \underline{x}_S = 0$, which implies

$$B = 0 \qquad (5.4)$$

The linear and angular momenta are $\underline{\xi} = \underline{\xi}_L + \underline{\xi}_S$ and $\underline{\eta} = \underline{\eta}_L + \underline{\eta}_S$. In accordance with the results of the preceding sections

$$\underline{\xi} = \partial T/\partial \underline{u} = M\underline{u} \qquad (5.5)$$

$$\underline{\eta} = \partial T/\partial \underline{w} + \Gamma \underline{J}^* = C\underline{w} + \Gamma \underline{J}^* \qquad (5.6)$$

If $\underline{F}$ and $\underline{L}$ are the resultants of the external forces and moments acting on the composite system, then the equations of motion become

$$M(\underline{u}' + \underline{w} \times \underline{u}) = \underline{F} \qquad (5.7)$$

$$C\, \underline{w}' + \underline{w} \times (C\, \underline{w} + \Gamma \underline{J}^*) = \underline{L} \qquad (5.8)$$

where ' denotes d/dt. Note that C is entirely determined by the geometry of the system and its mass distribution. The circulation of the fluid manifests itself only in the term $\Gamma \underline{J}^*$ of (5.8).

6. MOTION SUBJECT TO NO EXTERNAL TORQUE

The motion of a rigid body in the absence of external torques is a standard topic for mechanics textbooks[1]. Accordingly, the case $\underline{L} = 0$, which occurs for example for motion in a uniform gravitational field, should be ideally suited to bring out very clearly the novelties introduced by the inclusion of circulation.

For $\underline{L} = 0$ (5.8) becomes

$$C\,\underline{w}' + \underline{w} \times (C\,\underline{w} + \Gamma\,\underline{J}^*) = 0 \qquad (6.1)$$

Accordingly it is possible to determine the angular velocity $\underline{w}(t)$ independently of $\underline{u}(t)$. This system has two well-known integrals

$$\underline{w}^T C\,\underline{w} = 2T^* \qquad (6.2)$$

and

$$(C\,\underline{w} + \Gamma\,\underline{J}^*)^T\,(C\,\underline{w} + \Gamma\,\underline{J}^*) = K \qquad (6.3)$$

where the constant T^* is that part of the kinetic energy associated with $\underline{w}$, and the constant K is the square of the magnitude of the angular momentum vector. If the rectangular components of $\underline{w}$ are interpreted as rectangular coordinates, (6.2) is an ellipsoid with center at $\underline{w} = 0$, and (6.3) is an ellipsoid with center at $\underline{w} = -\Gamma C^{-1}\underline{J}^*$ which is in general different from $\underline{w} = 0$. Thus the integral curves of (6.1) are intersections of ellipsoids, but by contrast with the more familiar case $\Gamma = 0$, the centers of the ellipsoids no longer coincide.

It should be remarked that under the transformation $\underline{w}(t) = \Gamma\,\underline{\Omega}(\Gamma t)$ (6.1) takes the form

$$C d\underline{\Omega}/d\,\Gamma t + \underline{\Omega} \times (C\underline{\Omega} + \underline{J}^*) = 0$$

Thus, if the structure of the solutions of (6.1) has been determined for one value of $\Gamma \neq 0$, then it is known for all $\Gamma \neq 0$. We shall not exploit this fact, however, in the sequel.

Hereafter, let us suppose that T^* has been prescribed. To gain a comprehensive view of the behavior of the associated one-parameter family of integral curves for given Γ, let us consider the level curves of the function $K(\underline{w})$, defined by the left member of (6.3), on the energy ellipsoid (6.2). Let us begin by determining the critical points on (6.2) for which $K(\underline{w})$ is stationary. Proceed by Lagrange's method of undetermined multipliers. Let

$$F(\underline{w}, \lambda) = (C\,\underline{w} + \Gamma J^*)^T (C\,\underline{w} + \Gamma J^*) - \lambda(\underline{w}^T C \underline{w} - 2T^*)$$

Then at the desired critical points

$$\partial F/\partial \underline{w} = 2C^T (C\underline{w} + \Gamma \underline{J}^*) - 2\lambda\, C\, \underline{w} = 0$$

and $\partial F/\partial\lambda = 0$, which merely reasserts (6.2). Now by (5.2) $C^T = C$. Since, furthermore, C is non-singular, then $\partial F/\partial \underline{w} = 0$ implies

$$(C - \lambda\, I)\, \underline{w} = - \Gamma \underline{J}^* \tag{6.4}$$

If λ is not a characteristic root of C, then the critical points of $K(\underline{w})$ are defined by

$$\underline{w}_C = - (C - \lambda\, I)^{-1}\, \Gamma\, \underline{J}^* \tag{6.5}$$

as functions of λ. Of course these are merely the singular points of the system (6.1). Then (6.2) and (6.5) imply

$$\underline{J}^{*T} (C - \lambda\, I)^{-1} C (C - \lambda\, I)^{-1} \underline{J}^* = 2T^*/\Gamma^2 \tag{6.6}$$

from which λ must be determined. In accordance with (5.1) T^*/Γ^2 is proportional to the ratio of the kinetic energy due to $\underline{w}$ to that due to Γ

To simplify our discussion, let us assume that the $\underline{x}'$ coordinate axes have been chosen to be parallel to the principal axes of C. Then

$$C = \begin{pmatrix} D & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & F \end{pmatrix}$$

and, in general, we may assume

$$0 < D < E < F \tag{6.7}$$

If we also assume $\underline{J}^{*T} = (j,\ k,\ \ell)$, then (6.5) and (6.6) take the special forms

$$\underline{w}_c = (j/\lambda\text{-}D,\quad k/\lambda\text{-}E,\quad \ell/\lambda\text{-}F) \tag{6.8}$$

and

$$Dj^2(\lambda-D)^{-2} + Ek^2\ (\lambda-E)^{-2} + F\ell^2\ (\lambda-F)^{-2} = 2T^*/\Gamma^2 \tag{6.9}$$

If in (6.9) we consider T^*/Γ^2 to be a function of λ, then since $d^2(T^*\Gamma^{-2})/d\lambda^2 > 0$ always, $d(T^*\Gamma^{-2})/d\lambda$ always increases. Accordingly the graph of $T^*\Gamma^{-2}$ has the qualitative form shown in Figure 1, with the horizontal asymptote $T^*\Gamma^{-2} = 0$ and three vertical asymptotes $\lambda = D$, E, or F. Let $T^*\Gamma_1^{-2}$ and $T^*\Gamma_2^{-2}$ be relative minima of $T^*\Gamma^{-2}$, and assume $T^*\Gamma_1^{-2} < T^*\Gamma_2^{-2}$, though the sense of this inequality is not essential in the sequel. Then, depending on the size of $T^*\Gamma^{-2}$, (6.9) will have from two to six real roots. From Figure 1 it is apparent that the smallest real root will always be less than D, and the largest always exceeds F. In the limiting case $\Gamma^2 = 0$, we obtain three double roots $\lambda = D$, E, and F, which by (6.4) correspond to the ends of the principal axes of the energy ellipsoid (6.2), as indicated in Figure 1.

The general nature of the stationary values $K(\underline{w}_c)$ can be determined as follows. Let N_M, N_m, and N_s, respectively be the number of relative maxima, relative minima, and saddle points of K on (6.2). If these numbers are finite, the Morse theory of critical points[4] asserts

$$\begin{array}{ll} N_m \geqq 1 & N_M \geqq 1 \\ N_m - N_s \leqq 1, & N_M - N_s \leqq 1 \\ N_m + N_M - N_s = 2 & \end{array} \tag{6.10}$$

If we disregard occurrences of double roots λ, which will be taken into account later, (6.10) yields the possibilities tabulated hereafter:

	I	II	III
Critical points	6	4	2
$N_M + N_m$	4	3	2
N_s	2	1	0

When there are only two critical points, of course $N_m = N_M = 1$.

In the classical cases of rigid body motion or of motion of liquid filled bodies without circulation, K has two maxima at the ends of the minor axis of the energy ellipsoid, two minima at the ends of the major axis, and two saddle-points at the ends of the mean axis. K has the same value K_s at both saddle points, and the level curve $K = K_s$ divides the surface of the energy ellipsoid into four parts, each of which contains a family of closed integral (or level) curves surrounding one of the maxima or minima. Figure 2a is a schematic representation of the system of integral curves on a cut and flattened ellipsoid.

As Γ varies continuously the locations of the critical points will vary continuously on (6.2) as long as $0 \leqq \Gamma^2 \; \Gamma_2^2$. Continuous dependence on Γ will assure that maxima of K move into maxima, minima into minima, and saddle-points into saddle-points. For small $\Gamma^2 > 0$, however, the values of K at the two saddle points must differ. To show this, observe that by (6.3) and (6.4) we have

$$K = \lambda^2 \underline{w}_c^2 = K^*$$

at critical points, and then by (6.8) and (6.9)

$$K^*(\lambda) = \lambda^2 \left[\frac{2T^*}{E\,\Gamma^2} + (1 - \frac{D}{E}) \frac{j^2}{(\lambda - D)^2} - (\frac{F}{E} - 1) \frac{\ell^2}{(\lambda - F)^2} \right]$$

Then

$$\frac{dK^*}{d\lambda} = 2\lambda \left[\frac{2T^*}{E\,\Gamma^2} - (1 - \frac{D}{E}) \frac{j^2 D}{(\lambda - D)^3} + (\frac{F}{E} - 1) \frac{\ell^2 F}{(\lambda - F)^3} \right] \qquad (6.11)$$

Let $2\epsilon = \min(E-D, F-E)$. Then for some $\Gamma_\epsilon^2 \leqq \min(\Gamma_1^2, \Gamma_2^2)$ we shall have $\lambda^{-1} dK^*/d\lambda > 0$ for all $\Gamma^2 < \Gamma_\epsilon^2$ and $|\lambda - E| \leqq \epsilon$. Furthermore, there exists some $\Gamma_o^2 \leqq \Gamma_\epsilon^2$ such that (6.9) will have exactly two roots in $|\lambda - E| < \epsilon$ for $\Gamma^2 < \Gamma_o^2$. Since $dK^*/d\lambda > 0$ on the interval joining these roots, this implies that for $0 < \Gamma^2 < \Gamma_o^2$ the values of K^*, say K_1 and K_2, are different.

Now the level curves $K = K_1$ and $K = K_2 \neq K_1$ must continue to resemble lemniscates, with double points at the saddle points. Since they cannot intersect, the situation for small Γ^2 must resemble that shown in Figure 2b. The four families of integral curves surrounding a maximum or minimum within one of the lobes of $K = K_1$ or K_2 are obviously counterparts of families encountered for $\Gamma = 0$. The novelty introduced for small $\Gamma^2 > 0$ is the occurrence of a fifth set of closed integral curves, typified by the dashed curve between the level curves $K = K_1$ and $K = K_2$. Although our proof that $K_1 \neq K_2$ is valid only for sufficiently small values of Γ^2, it seems plausible that the result is true for $0 < \Gamma^2 < \Gamma_2^2$.

As suggested by Figure 1, when $\Gamma^2 = \Gamma_2^2$ one of the maxima (for the conditions depicted in our graph) should coalesce with the saddle-point S_2. For $\Gamma_2^2 < \Gamma^2 < \Gamma_1^2$, there remain two minima, one maximum, and one saddle-point. Now there will be three types of closed integral curves. When $\Gamma^2 = \Gamma_1^2$ one of the minima will coalesce with the remaining saddle-point. Finally, for $\Gamma_1^2 < \Gamma^2$, there remain one maximum and one minimum. Now there is only one set of closed integral curves. From Figure 1 it is clear that $|\lambda|$ and $|\Gamma|$ tend to infinity together. Since for large $|\lambda|$ we can expand

$$(C - \lambda I)^{-1} = -\lambda^{-1}\left(I + \sum_1^\infty C^n \lambda^{-n}\right)$$

then by (6.6)

$$J^{*T}\left[I + \sum C^n \lambda^{-n}\right] C \left[I + \sum C^n \lambda^{-n}\right] \underline{J} = 2T^*\lambda^2/\Gamma^2$$

Hence

$$\lim_{|\Gamma| = \infty} (\Gamma/\lambda)^2 = 2T^*/\underline{J}^{*T}C\underline{J}^*$$

Then by (6.5)

$$\lim_{|\Gamma| = \infty} \underline{w}_c = \pm (2T^*/\underline{J}^{*T}C\underline{J}^*)^{0.5}\underline{J}^*$$

i.e., the critical points for $|\Gamma| = \infty$ are at the ends of the diameter of the energy ellipsoid parallel to J^*.

In the classical case, $\Gamma = 0$, the components of $\underline{w}$ can be expressed in terms of elliptic functions. For $\Gamma \neq 0$ and D, E, F all distinct this no longer appears to be the case. For, let us attempt to express w_y and w_z as functions of w_x on an integral curve. When we eliminate w_z, for example from (6.2) and (6.3), we shall obtain, in general an equation involving a polynomial of fourth degree in w_y, in which all powers of w_y between zero and four can actually occur. Thus w_y becomes a complicated algebraic function of w_x, and so, presumably, does w_z. Thus integration of the equation of motion that involves w_x will presumably lead to something more complicated than elliptic functions. However, Dr. Č. Masaitis has observed that if $\underline{J}^*$ is parallel to a principal axis of C then $\underline{w}$ is expressible in terms of elliptic functions.

7. DEGENERATE CASES

Axisymmetric Systems

In Section 6 we assumed that all characteristic roots of the matrix C were distinct, and that $\underline{J}^*$ was parallel to none of the principal axes of C. If $D \neq E = F$ and $j^2(k^2 + \ell^2) \neq 0$, for example, then (6.9) takes the degenerate form

$$\frac{Dj^2}{(\lambda-D)^2} + \frac{E(k^2 + \ell^2)}{(\lambda-E)^2} = 2T^*/\Gamma^2 \tag{7.1}$$

We obtain a similar equation for the determination of λ if we assume that D, E, F are distinct, but exactly one of the components of $\underline{J}^*$ vanishes. Now for small $|\Gamma|$ the analysis starts with four critical points, one of which must be a saddle-point. With increasing $|\Gamma|$ we pass finally to two critical points, just as in the more general circumstances considered in Section 6.

If D = E = F, then (6.6) degenerates to

$$D\underline{J}^{*2}/(\lambda-D)^2 = 2T^*/\Gamma^2 \tag{7.2}$$

We obtain a similar equation, regardless of the nature of the characteristic roots of C, if only one component of $\underline{J}^*$ is non-zero, or if E = F and j = 0. A complete enumeration of the possibilities has no especial interest, and in any event, all types that can arise have been mentioned already. Now there are always only two critical points.

To turn to the most important of these degenerate cases, suppose our liquid-solid system is axisymmetric, with respect to the x-axis. Much more can now be said about form of the velocity potential. Let us introduce cylindrical polar coordinates

$$x = x, \quad y + iz = r\, e^{i\theta}$$

If we let n_r be the radial component of the inward normal to S_L, then

$$n_y + in_z = n_r e^{i\theta}$$

Since now $(\underline{x} \times \underline{n})_x = 0$, we observe that

$$\sigma_x = 0 \tag{7.3}$$

satisfies the relevant parts of equations (2.5) and (2.6). Since

$$(\underline{x} \times \underline{n})_z - i(\underline{x} \times \underline{n})_y = (n_r x - n_x r)\, e^{i\theta}$$

$$\sigma_z(\underline{x}) - i\sigma_y(\underline{x}) = \psi(x,r)\, e^{i\theta} \tag{7.4}$$

will satisfy (2.5) and (2.6) if

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} = 0 \qquad \text{in } S_L^* \tag{7.5}$$

$$n_x \frac{\partial\psi}{\partial x} + n_r \frac{\partial\psi}{\partial r} = n_r x - n_x r \qquad \text{on } C_L^* \tag{7.6}$$

where S_L^* is a cross section of V_L in any plane θ = constant, and C* is its boundary. Obviously

$$\tau(\underline{x}) = \theta/2\pi \tag{7.7}$$

satisfies (2.9) and (2.10) and increases by unity for each positively-directed circuit of a circle $x = x_o$, $r = r_o$.

To write the equations of motion we would require

$$2T^* = \underline{w}^T C_L\, \underline{w} = \int_{V_L} \rho_L\, (\nabla\, \underline{w} \cdot \underline{\sigma})^2\, dV$$

By (7.3) and (7.4) this becomes

$$2T^* = \pi\rho_L\, (w_y^2 + w_z^2) \iint_{S_L^*} \left[\psi_x^2 + \psi_r^2 + \psi^2 r^{-2} \right] r\, dr\, dx \tag{7.8}$$

By means of (7.5) and (7.6), and Gauss' theorem this can also be written in the more convenient form for calculation

$$2T^* = -\,\pi\rho_L(w_y^2 + w_z^2) \int_{C_L^*} r\psi\, (n_r x - n_x r)\, ds \tag{7.9}$$

where s is arc-length along C_L^*. Clearly the matrix C_L is proportional to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We also need $\underline{J}^*$. Since for (7.7)

$$-2\pi\ (\underline{x} \times \nabla \tau) = (-1,\ x \cos \theta/r,\ \ x \sin \theta/r)$$

then $\underline{J}^* = -\int_{V_L} \rho_L\, \underline{x} \times \nabla \tau \, dV$ yields

$$-\ \underline{J}^* = (\rho_L \iint_{S_L^*} r\, dr dx,\ 0,\ 0) \tag{7.10}$$

It should also be remarked that the transverse moments of inertia of the solid $E_S = F_S$, i.e.

$$C_S = \begin{pmatrix} D_S & 0 & 0 \\ 0 & E_S & 0 \\ 0 & 0 & E_S \end{pmatrix}$$

Thus $C = C_S + C_L$ will also be diagonal, and the last two diagonal elements will be equal.

Now (6.2) and (6.3) become

$$D\, w_x^2 + E\ (w_y^2 + w_z^2) = 2T^* \tag{7.11}$$

$$(D\, w_x + \Gamma\, j)^2 + E^2 (w_y^2 + w_z^2) = K \tag{7.12}$$

where j is the x-component of $\underline{J}^*$ in (7.8). The intersections of these ellipsoids of revolutions are circles

$$w_x = \text{constant}$$

$$w_y^2 + w_z^2 = (2T^* - D\, w_x^2)/E \equiv R^2$$

Thus the solutions of the equations of motion (6.1) become

$$
\begin{aligned}
w_x &= \text{constant} \\
w_y + i w_z &= R\, e^{i\nu\,(t-t_0)}
\end{aligned}
\tag{7.13}
$$

where the rate of precession

$$\nu = (1 - D/E) w_x - \Gamma j/E \tag{7.14}$$

For practical purposes one would certainly be primarily interested in axisymmetric systems. However, it should be remarked that small errors will occur in machining models intended to be axisymmetric. If many models are to be constructed, one may also deliberately abandon nearly perfect dynamic axisymmetry because the effort to achieve it does not result in sufficient improvement in the performance of the model. Therefore the more general analysis of Section 6 may also have some bearing on practical applications. For systems that are not quite axisymmetric one would expect the full sequence of possibilities depicted in Figure 2 to occur. In Figures 2a and 2b, however, the areas occupied by closed integral curves surrounding M_1 and m_1 would presumably cover most of the surface of the energy ellipsoid, and the remaining sets of integral curves should be cramped into a relatively small fraction of the surface area.

8. TOPOLOGICALLY COMPLICATED CAVITIES

Let us motivate the following discussion by beginning with a special configuration. Consider a solid with a toroidal cavity V_L bounded by the concentric cylinders $r = r_1 \pm r_o$ and the parallel planes $x = \pm x_1$. To modify the motion of the solid-liquid system let us insert partitions on the planes $\theta = 2\pi m/n$, $1 \leqq m \leqq n$. This will merely subdivide the cavity into n topologically spherical regions, which creates no dynamical novelty. Suppose, however, that all of the partitions are perforated. If each partition contains exactly one hole, V_L will be topologically toroidal (i.e. homeomorphic to the interior of a torus). But if we suppose that some of the partitions contain more than one hole the topological structure of V_L will become more complicated than the interior of a torus. Since the following discussion will not be concerned with the exact number of perforations, we shall emphasize its generality by suggesting that the partitions could even be imagined to be made of finely woven wire mesh, to impart an extremely complicated topological structure to V_L.

Now let us indicate how the treatment of the toroidal cavity can be adapted to the case of a very general cavity V_L, not necessarily constructed by the process described in the preceding paragraph. In our earlier discussion, the topological nature of V_L asserted itself through applications of Gauss' theorem to various volume integrals taken over V_L. Let us suppose that V_L is bounded by a two-sided surface of genus N. In other words, V_L can be considered to be topologically equivalent to the interior of a sphere with N handles. Then, by the insertion of N partitions S_j^* we can make V_L into a topological sphere V_L'. With each partition S_j^* we associate a velocity potential function $\tau_j(\underline{x})$ which produces unit circulation on a closed path in V_L' from the (arbitrarily chosen) initial side to the final side of S_j^*. In fact, τ_j will be single-valued in the cavity of genus N-1 produced by inserting only the partition S_j^* but none of the others. Now replace (2.8) by [2]

$$\Phi\,(\underline{x},t) = \underline{x}\cdot\underline{u}(t) + \underline{w}(t)\cdot\underline{\sigma}\,(\underline{x}) + \sum \Gamma_j \tau_j(\underline{x}) \qquad (8.1)$$

where Γ_j is a constant circulation associated with τ_j. The boundary conditions for $\underline{\sigma}$ ($\underline{x}$) are as before, while, analogously to (2.10)

$$\partial\tau_j/\partial n = 0 \qquad \text{on } S_L \tag{8.2}$$

The kinetic energy of the solid-liquid system becomes

$$2T = M\underline{u}^2 + \underline{w}^T C\, \underline{w} + \sum A_{ij}\, \Gamma_i\, \Gamma_j \tag{8.3}$$

where

$$A_{ij} = \int_{V_L} \rho_L\, \nabla\tau_i \cdot \nabla\tau_j\, dV \tag{8.4}$$

is a constant positive-definite matrix. As before, the linear momentum is

$$\underline{\xi} = \partial T/\partial\underline{u} = M\,\underline{u} \tag{8.5}$$

and the angular momentum

$$\underline{\eta} = \partial T/\partial\underline{w} + \sum \Gamma_j \underline{J}^*_j \tag{8.6}$$

where the <u>axis vector</u>

$$\underline{J}^*_j = \int_{V_L} \rho_L\, \underline{x} \times \nabla\tau_j\, dV \tag{8.7}$$

is associated with τ_j.

For motion subject to no external moment we again obtain the energy integral

$$\underline{w}^T C\underline{w} = 2T^* \tag{8.8}$$

and the angular momentum integral

$$(C\underline{w} + \sum \Gamma_j\, \underline{J}^*_j)^2 = K \tag{8.9}$$

The search for the critical points, $\underline{w}_c$, of the function K on the energy ellipsoid (8.8) leads to

$$C\underline{w}_c + \sum \Gamma_j \underline{J}^*_j = \lambda\underline{w}_c \tag{8.10}$$

or

$$\underline{w}_c = -(C - \lambda I)^{-1} \quad \sum \Gamma_j \underline{J}^*_j \tag{8.11}$$

When we substitute (8.11) into (8.8) we obtain as an analog of (6.6)

$$\sum \Gamma_j \underline{J}^{*T}_j \, P(\lambda) \quad \sum \Gamma_j \underline{J}^*_j = 2T^* \tag{8.12}$$

where

$$P(\lambda) = (C - \lambda I)^{-1} \, C(C - \lambda I)^{-1} \tag{8.13}$$

The analysis of the nature of the integral curves as a function of the N parameters Γ_j can be carried out along the following lines. Suppose that $n(\leqq N)$ of the vectors $\underline{J}^*_j$ are linearly independent, where $1 \leqq n \leqq 3$. Let $\underline{H}_\alpha$, for $1 \leqq \alpha \leqq n$, be an orthonormal basis for the set $\underline{J}^*_j$. Then we must have

$$\underline{H}^2_\alpha = 1, \qquad \underline{H}_\alpha \cdot \underline{H}_\beta = 0, \qquad \alpha \neq \beta \tag{8.14}$$

Also, there must exist an N x n matrix $G_{j\alpha}$ of constant elements, and of rank n, such that

$$\underline{J}^*_j = \sum_1^n G_{j\alpha} \underline{H}_\alpha \tag{8.15}$$

Now

$$\sum_1^N \Gamma_j \underline{J}^*_j = \sum_1^n \left(\sum_1^N \Gamma_j G_{j\alpha} \right) \underline{H}_\alpha$$

Let W_α be any n-dimensional unit vector, i.e.

$$\sum_1^n W^2_\alpha = 1 \tag{8.16}$$

There is, of course, an n-1 parameter family of W_α. Then for any Γ the system of linear equations

$$\sum_1^N \Gamma_j G_{j\alpha} = \Gamma W_\alpha \tag{8.17}$$

has an (N-n)-parameter family of solutions Γ_j. The general solution of (8.17) is of the form

$$\Gamma_j = \Gamma\left(\gamma_{jo} + \sum_1^{N-n} A_\epsilon \gamma_{j\epsilon}\right) \qquad \text{if } \Gamma \neq 0$$
$$\Gamma_j = \sum_1^{N-n} A_\epsilon \gamma_{j\epsilon} \qquad \text{if } \Gamma = 0 \tag{8.18}$$

where γ_{jo} is a particular solution of

$$\sum \gamma_{jo} G_{j\alpha} = W_\alpha$$

and if $N > n$, then $\gamma_{j\epsilon}$ are N-n linearly independent solutions of the associated homogeneous equations, and A_ϵ are N-n arbitrary constants. Now

$$\sum_1^N \Gamma_j \underline{J}^*_j = \Gamma \sum_1^n W_\alpha \underline{H}_\alpha \tag{8.19}$$

and (8.12) takes the form

$$\sum_1^n W_\alpha \underline{H}_\alpha^T P(\lambda) \sum_1^n W_\alpha \underline{H}_\alpha = 2T^*/\Gamma^2 \qquad \text{if } \Gamma \neq 0 \tag{8.20}$$

which is now more closely analogous to (6.6). The kinetic energy due to circulation is

$$0.5 \sum A_{ij} \Gamma_i \Gamma_j = \begin{cases} \Gamma^2 c^2 & \text{if } \Gamma \neq 0 \\ d^2 & \text{if } \Gamma = 0 \end{cases} \tag{8.21}$$

where

$$c^2 = 0.5 \sum A_{ij} \left(\gamma_{io} + \sum_1^{N-n} A_\epsilon \gamma_{i\epsilon}\right)\left(\gamma_{jo} + \sum_1^{N-n} A_\theta \gamma_{j\theta}\right)$$
$$d^2 = 0.5 \sum A_{ij} \sum_1^{N-n} A_\epsilon \gamma_{i\epsilon} \sum_1^{N-n} A_\theta \gamma_{j\theta} \tag{8.22}$$

Thus $T^*/\Gamma^2 c^2$ would be the ratio of kinetic energy due to $\underline{w}$ to that due to circulation when $\Gamma \neq 0$. Note that if $N > n$, then c^2 and d^2 may vary with the choice of $\mathbf{A}_\epsilon$.

If we make particular choices of W_α and c^2, then in general we have exactly the relation between Γ and the integral curves on the energy ellipsoid that is described in Section 6. If N-n $\geqq$ 2, (8.22) has an (N-n-1) - parameter family of solutions A_ϵ. This simply means that the same set of motions of the system can be realized with an (N-n-1) - parameter set of choices of the circulations Γ_j.

Since equation (8.20) is the crucial element in the discussion of possible motions of the solid-liquid system, then the categorization of motions should clearly be based on whether $\Gamma = 0$ or $\Gamma \neq 0$, and then, in the latter case, on the number of parameters, n-1, required to determine W_α. Thus there will be four major types of motion. The distinctions between them could conceivably be visualized and clarified by describing some of their properties, such as (1) the locus of the centers

$$\underline{w} = - C^{-1} \sum \Gamma_j \underline{J}_j^* = - \Gamma \sum_1^n W_\alpha C^{-1} \underline{H}_\alpha \tag{8.23}$$

of the angular momentum ellipsoids (8.9) as a function of Γ and W_α; (2) the possible steady states of rotation for fixed Γ and (if possible) variable W_α; and (3) the limiting steady states for $|\Gamma| = \infty$.

CASE I. If $\Gamma = 0$, then by (8.23) the center of the angular momentum ellipsoid is <u>at the origin</u>. If $N > n$ there may actually be circuits with circulation $\Gamma_j \neq 0$, in accordance with the second part of (8.18). By (8.19) for $\Gamma=0$ and (8.10) λ must be a characteristic root of C, and $\underline{w}_c$ a characteristic vector. This leads to the familiar rigid body and spherical cavity type of motion.

CASE II. If $\Gamma \neq 0$ and n = 1, the results of Section 6 are applicable word for word. By (8.23) the centers of the momentum ellipsoids are on a <u>straight line</u> through the origin parallel to $C^{-1}\underline{H}_1$. For fixed Γ there are from two to six possible steady states of rotation, depending on the magnitude of Γ. For $|\Gamma| = \infty$ there are two steady states of rotation at the ends of the diameter of the energy ellipsoid parallel to the vector $C^{-1}\underline{H}_1$. Such motions occur, in particular, for toroidal cavities.

CASE III. If $\Gamma \neq 0$ and $n = 2$, by (8.23) the centers of the angular momentum ellipsoids are on a plane through the origin. Let us set $W_1 = \cos \Theta$, $W_2 = \sin \Theta$. For fixed Γ and Θ there are from two to six steady states of rotation, depending on the value of Γ. If Θ varies while Γ remains fixed the critical points

$$\underline{w}_c = - \Gamma \left[C - \lambda(\Theta, \Gamma) I \right]^{-1} (\cos \Theta \, \underline{H}_1 + \sin \Theta \, \underline{H}_2)$$

will trace a set of curves on the energy ellipsoid. In accordance with the results obtained at the end of Section 6, for $|\Gamma| = \infty$ and fixed Θ the two corresponding steady states of rotation will be at the ends of the diameter parallel to $\sum W_\alpha \underline{H}_\alpha = \cos \Theta \, \underline{H}_1 + \sin \Theta \, \underline{H}_2$. In other words, for $|\Gamma| = \infty$ the possible steady states are on the intersection of the energy ellipsoid and the plane $\underline{H}_1 \times \underline{H}_2 \cdot \underline{w} = 0$.

CASE IV. If $\Gamma \neq 0$ and $n = 3$, then the centers of the angular momentum ellipsoids can be anywhere in $\underline{w}$-space. Let us set $W_1 = \cos \phi \cos \Theta$, $W_2 = \cos \phi \sin \Theta$, $W_3 = \sin \Theta$. For fixed Γ, ϕ and Θ, there will be from two to six steady states of rotation, depending on the value of Γ. If ϕ and Θ vary independently while Γ remains fixed, the critical points

$$\underline{w}_c = - \Gamma \left[C - \lambda (\phi, \Theta, \Gamma) I \right]^{-1} (\cos \phi \cos \Theta \, \underline{H}_1 + \cos \phi \sin \Theta \, \underline{H}_2 + \sin \phi \, \underline{H}_3)$$

will trace out regions on the energy ellipsoid. In accordance with the limits calculated at the end of Section 6, for fixed ϕ and Θ the two steady states of rotation for $|\Gamma| = \infty$ will be at the ends of the diameter of the energy ellipsoid parallel to

$$\cos \phi \, (\cos \Theta \, \underline{H}_1 + \sin \Theta \, \underline{H}_2) + \sin \phi \, \underline{H}_3$$

If we let ϕ and Θ range over all permissible values, we obtain the entire surface of the energy ellipsoid.

Let us conclude by reiterating that Case I includes the topologically spherical cavity (genus zero), and Case II the toroidal cavity (genus one). It certainly seems reasonable to conjecture that Cases III and IV, respectively, correspond to cavities, or at least to some cavities, of genera

two and three, respectively. In a cavity of genus N it is possible to assign arbitrarily N independent circulations Γ_j . For $N \geqq 4$ our conjecture would imply that increases in genus do not lead to new types of dynamic behavior, but merely present a greater variety of choices of parameters (Γ_j) to simulate the behavior of bodies with liquid filled cavities of genera less than four. If there were only some mechanism for randomly exciting and varying strong circulations in a body with a complicated, liquid filled labyrinth, one might speculate that then by virtue of the possibility of suddenly inducing degeneracies of the sort discussed in Section 7, and thereby switching from motion of one type to another, the behavior of the liquid-solid system could become highly erratic and unstable.

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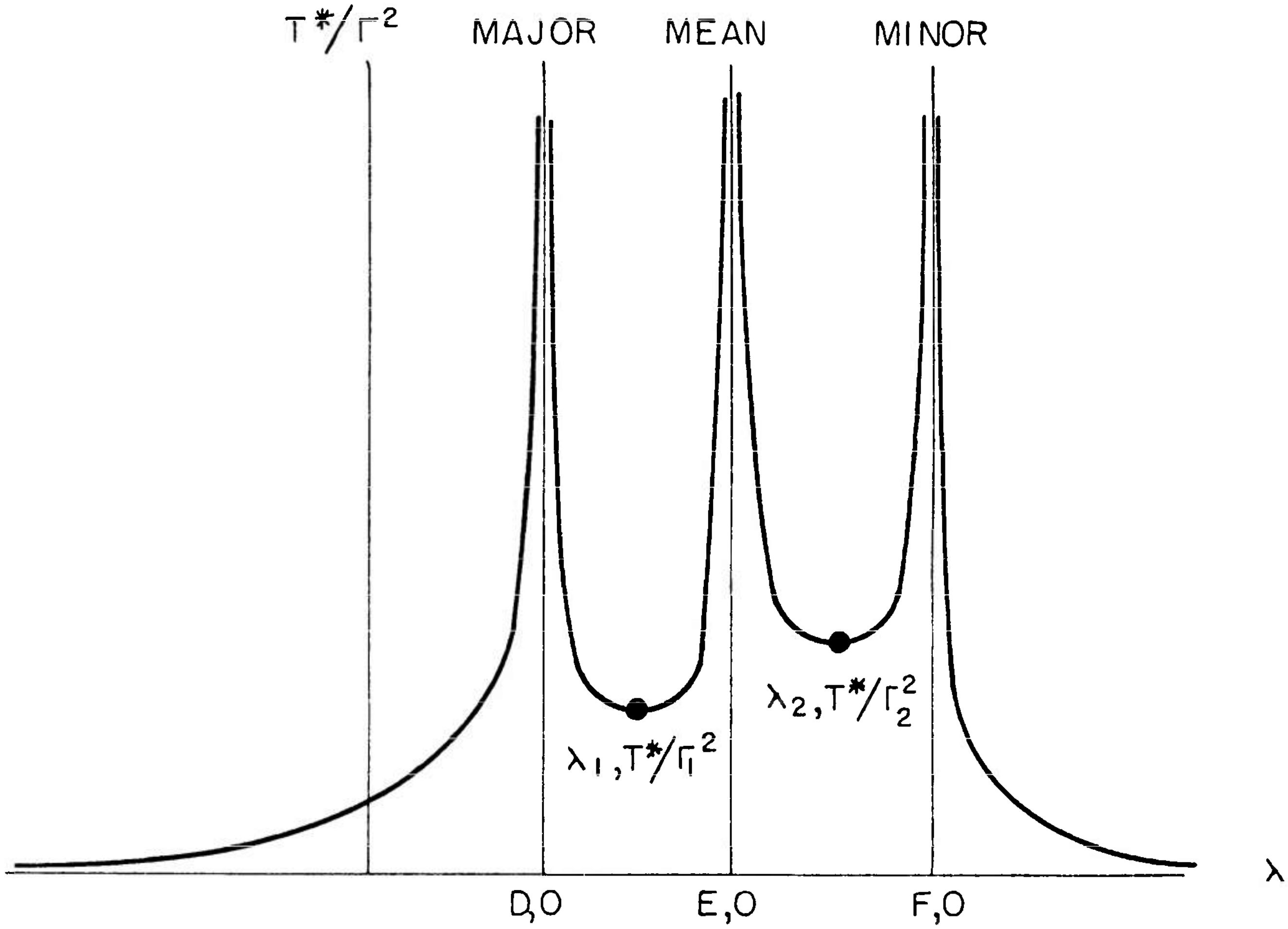

FIGURE I KINETIC ENERGY RATIO VERSUS λ

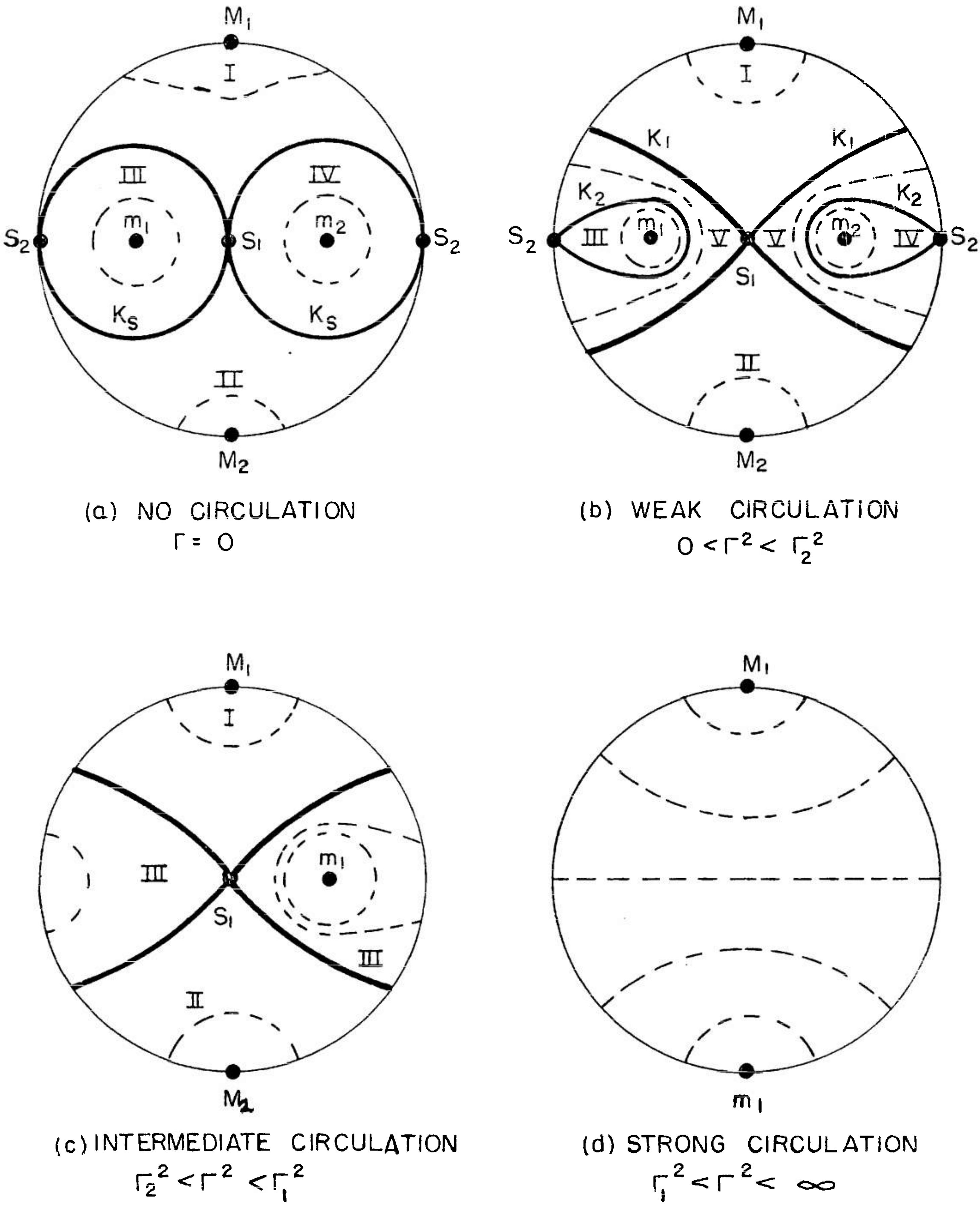

FIGURE 2: CLASSIFICATION OF INTEGRAL CURVES AS A FUNCTION OF CIRCULATION

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